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Degradation of Entanglement in Moving Frames

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Abstract The distillability of bipartite entangled state as seen by moving observers has been investigated. It is found that the same initial entanglement for a state parameter α and its "normalized partner" $\sqrt{1-\alpha^2}$ will be degraded as seen by moving observer. It is shown that in the ultra relativistic limit, the state does not have distillable entanglement for any α.

Keywords Lorentz transformation · Entanglement

Relationship between special relativity and quantum information theory is discussed by many authors [1]. Peres et al. [2] investigated the relativistic properties of spin entropy for a single, free particle of spin$-1/2$. They show that the usual definition of quantum entropy has no invariant meaning in special relativity. The reason is that, under a Lorentz boost, the spin undergoes a Wigner rotation whose direction and magnitude depend on the momentum of the particle. Even if the initial state is a direct product of a function of momentum and a function of spin, the transformed state is not a direct product. Lamata et al. [3] define weak and strong criteria for relativistic isoentangled and isodistillable states to characterize relative and invariant behavior of entanglement and distillability. In a recent paper [4], author shows how violation of Bell's inequality is frame dependent. In this letter, we choose a generic state as the initial entangled state and we will try to show that the entanglement is degraded as seen by the relativistically observer. This help us to understand the relationship between special relativity and quantum information theory. The initial nonmaximal entangled state is

$$|\Phi\rangle = \alpha\Psi_1^{(a)}(\mathbf{p}_a)\Psi_1^{(b)}(\mathbf{p}_b) + \sqrt{1-\alpha^2}\Psi_2^{(a)}(\mathbf{p}_a)\Psi_2^{(b)}(\mathbf{p}_b), \tag{1}$$

where α is some number that satisfies $|\alpha| \in (0,1)$. Here $\mathbf{p}_a$ and $\mathbf{p}_b$ are the corresponding momentums vectors of particles A and B and

$$\Psi_1^{(a)}(\mathbf{p}_a) = g(\mathbf{p}_a)|0\rangle = |0, \mathbf{p}_a\rangle = \begin{pmatrix} g(\mathbf{p}_a) \\ 0 \end{pmatrix}, \tag{2}$$

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$$\Psi_2^{(a)}(\mathbf{p}_a) = g(\mathbf{p}_a)|1\rangle = |1, \mathbf{p}_b\rangle = \begin{pmatrix} 0 \\ g(\mathbf{p}_a) \end{pmatrix}. \tag{3}$$

For simplicity assume that they are sufficiently well localized around momenta $\mathbf{p}$, under the Lorentz transformation the states (2) and (3) transformed as [3]

$$\Lambda[\Psi_1(\mathbf{p})] = \begin{pmatrix} b_1(\mathbf{p}) \\ b_2(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \cos(\theta_{\mathbf{p}}/2) \\ \sin(\theta_{\mathbf{p}}/2) \end{pmatrix} g(\mathbf{p}) = \cos(\theta_{\mathbf{p}}/2)|0, \mathbf{p}\rangle + \sin(\theta_{\mathbf{p}}/2)|1, \mathbf{p}\rangle, \tag{4}$$

$$\Lambda[\Psi_2(\mathbf{p})] = \begin{pmatrix} -b_2(\mathbf{p}) \\ b_1(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} -\sin(\theta_{\mathbf{p}}/2) \\ \cos(\theta_{\mathbf{p}}/2) \end{pmatrix} g(\mathbf{p}) = -\sin(\theta_{\mathbf{p}}/2)|0, \mathbf{p}\rangle + \cos(\theta_{\mathbf{p}}/2)|1, \mathbf{p}\rangle, \tag{5}$$

where $\theta_{\mathbf{p}}$ is Wigner angle satisfies the relation

$$\tan\theta_{\mathbf{p}} = \frac{\sinh\xi \sinh\delta}{\cosh\xi + \cosh\delta}, \tag{6}$$

here $\cosh\xi = (1-\beta^2)^{-1/2}$ where β is boost speed and $\cosh\delta = p_0/m$. Now under Lorentz transformation the state transformed to (after tracing over momentum eigenket $|\mathbf{p}\rangle$),

$$\begin{aligned} |\Phi\rangle_\Lambda = {} & \left(\alpha\cos^2(\theta_{\mathbf{p}}/2) + \sqrt{1-\alpha^2}\sin^2(\theta_{\mathbf{p}}/2)\right)|00\rangle \\ & + \sin(\theta_{\mathbf{p}}/2)\cos(\theta_{\mathbf{p}}/2)(\alpha - \sqrt{1-\alpha^2})(|01\rangle + |10\rangle) \\ & + \left(\alpha\sin^2(\theta_{\mathbf{p}}/2) + \sqrt{1-\alpha^2}\cos^2(\theta_{\mathbf{p}}/2)\right)|11\rangle. \end{aligned} \tag{7}$$

A very popular measure for the quantification of bipartite quantum correlations is the concurrence [5]. This quantity can be defined

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{8}$$

with λ_i being the square roots of the eigenvalues of $\rho_{AB}(\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y)$ where the asterisk denotes complex conjugation and $\sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$. Now the concurrence for this state is

$$C(|\Phi\rangle_\Lambda) = 2\alpha\sqrt{1-\alpha^2}. \tag{9}$$

Which is the Lorentz invariant concurrence. To be more precise one should take wave packets in momentum space, with Gaussian momentum distributions $g(\mathbf{p}) = \pi^{-3/4}w^{-3/2}$ $\exp(-|\mathbf{p}|^2/2w^2)$. If we trace the momentum degrees of freedom we obtain the usual entangled state $|\phi\rangle = \alpha|00\rangle + \sqrt{1-\alpha^2}|11\rangle$. The general density matrix for two particle systems with momentums $\mathbf{p}_a$ and $\mathbf{p}_b$ is given by

$$\rho_\Phi = \sum_{ijkl=1,2} C_{ijkl}\Psi_i(\mathbf{p}_a) \otimes \Psi_j(\mathbf{p}_b)[\Psi_k(\mathbf{p}'_a) \otimes \Psi_l(\mathbf{p}'_b)]^\dagger. \tag{10}$$

For state (1) the coefficients C_{ijkl} are

$$\begin{aligned} & C_{1111} = \alpha^2, \qquad C_{2222} = 1-\alpha^2, \\ & C_{1122} = C_{2211} = \alpha\sqrt{1-\alpha^2} \end{aligned} \tag{11}$$

For obtaining the Lorentz transformation of (10), we need the relativistic properties of spin entropy for a single, free particle of spin-$1/2$. The quantum state of a spin-$1/2$ particle can be written in the momentum representation as follows

$$\Psi(\mathbf{p}) = \begin{pmatrix} a_1(\mathbf{p}) \\ a_2(\mathbf{p}) \end{pmatrix}, \tag{12}$$

where

$$\int (|a_1(\mathbf{p})|^2 + |a_2(\mathbf{p})|^2) d\mathbf{p} = 1. \tag{13}$$

The density matrix corresponding to state (12) is

$$\rho(\mathbf{p}', \mathbf{p}'') = \begin{pmatrix} a_1(\mathbf{p}')a_1(\mathbf{p}'')^* & a_1(\mathbf{p}')a_2(\mathbf{p}'')^* \\ a_1(\mathbf{p}')a_2(\mathbf{p}'')^* & a_2(\mathbf{p}')a_2(\mathbf{p}'')^* \end{pmatrix}. \tag{14}$$

By setting $\mathbf{p}' = \mathbf{p}'' = \mathbf{p}$ and integrating over $\mathbf{p}$ we obtain the reduced density matrix for spin

$$\sigma = \frac{1}{2} \begin{pmatrix} 1 + n_z & n_x - i n_y \\ n_x + i n_y & 1 - n_z \end{pmatrix}, \tag{15}$$

where the Bloch vector $\mathbf{n}$ is given by

$$n_z = \int (|a_1(\mathbf{p})|^2 - |a_2(\mathbf{p})|^2) d\mathbf{p}, \tag{16}$$

$$n_x - i n_y = \int a_1(\mathbf{p}) a_2(\mathbf{p})^* d\mathbf{p}. \tag{17}$$

Now under Lorentz boost density matrix (10) transformed into [3]

$$\begin{aligned} \Lambda \rho_\Phi \Lambda^\dagger = \sum_{ijkl=1,2} & C_{ijkl} \Lambda(p_a) \Psi_i(\mathbf{p}_a) \otimes \Lambda(p_b) \Psi_j(\mathbf{p}_b) \\ & \times [\Lambda(p_a)\Psi_k(\mathbf{p}'_a) \otimes \Lambda(p_b)\Psi_l(\mathbf{p}'_b)]^\dagger. \end{aligned} \tag{18}$$

The reduced density matrix for spin is obtained by setting $\mathbf{p}_a = \mathbf{p}'_a$, $\mathbf{p}_b = \mathbf{p}'_b$ and tracing over momentum

$$\begin{aligned} \tau = \mathrm{Tr}_{\mathbf{p}_a \mathbf{p}_b}[\Lambda \rho_\Phi \Lambda^\dagger] = \sum_{ijkl=1,2} & C_{ijkl} \, \mathrm{Tr}_{\mathbf{p}_a} \left\{ \Lambda(p_a)\Psi_i(\mathbf{p}_a)[\Lambda(p_a)\Psi_k(\mathbf{p}_a)]^\dagger \right\} \\ & \otimes \mathrm{Tr}_{\mathbf{p}_b} \left\{ \Lambda(p_b)\Psi_j(\mathbf{p}_b)[\Lambda(p_b)\Psi_l(\mathbf{p}_b)]^\dagger \right\}. \end{aligned} \tag{19}$$

To leading order $w/m \ll 1$ we have

$$n_z = n \approx 1 - \left(\frac{w}{2m} \tanh \frac{\xi}{2} \right)^2, \qquad n_x = n_y \approx 0. \tag{20}$$

It can be appreciated in (19) that the expression is decomposable in the sum of the tensor products of 2×2 spin blocks, each corresponding to each particle. We compute now the different blocks, corresponding to the four possible tensor products of the states (4) and (5):

$$\mathrm{Tr}_{\mathbf{p}}\left\{\Lambda(p)\Psi_1(\mathbf{p})[\Lambda(p)\Psi_l(\mathbf{p})]^{\dagger}\right\} = \frac{1}{2}\begin{pmatrix} 1+n & 0 \\ 0 & 1-n \end{pmatrix}, \tag{21}$$

$$\mathrm{Tr}_{\mathbf{p}}\left\{\Lambda(p)\Psi_2(\mathbf{p})[\Lambda(p)\Psi_2(\mathbf{p})]^{\dagger}\right\} = \frac{1}{2}\begin{pmatrix} 1-n & 0 \\ 0 & 1+n \end{pmatrix}, \tag{22}$$

$$\mathrm{Tr}_{\mathbf{p}}\left\{\Lambda(p)\Psi_1(\mathbf{p})[\Lambda(p)\Psi_2(\mathbf{p})]^{\dagger}\right\} = \frac{1}{2}\begin{pmatrix} 0 & 1+n \\ -(1-n) & 0 \end{pmatrix}, \tag{23}$$

$$\mathrm{Tr}_{\mathbf{p}}\left\{\Lambda(p)\Psi_2(\mathbf{p})[\Lambda(p)\Psi_1(\mathbf{p})]^{\dagger}\right\} = \frac{1}{2}\begin{pmatrix} 0 & -(1-n) \\ 1+n & 0 \end{pmatrix}. \tag{24}$$

With the help of (21)–(24), it is possible to compute the effects of the Lorentz transformation, associated with a boost in the x-direction, on any density matrix of two spin-$1/2$ particles with factorized Gaussian momentum distributions. In particular density matrix (19) is reduced to

$$\tau = \frac{1}{4}\begin{pmatrix} 4\alpha^2 n + (1-n)^2 & 0 & 0 & 2\alpha\sqrt{1-\alpha^2}(1+n^2) \\ 0 & 1-n^2 & -2\alpha\sqrt{1-\alpha^2}(1-n^2) & 0 \\ 0 & -2\alpha\sqrt{1-\alpha^2}(1-n^2) & 1-n^2 & 0 \\ 2\alpha\sqrt{1-\alpha^2}(1+n^2) & 0 & 0 & -4\alpha^2 n + (1+n)^2 \end{pmatrix}. \tag{25}$$

We can apply now the positive partial transpose criterion [6] to know whether this state is entangled and distillable. The partial transpose criterion provides a sufficient condition for the existence of entanglement in this case: if at least one eigenvalue of the partial transpose is negative, the density matrix is entangled; but a state with positive partial transpose can still be entangled. It is the well-known bound or nondistillable entanglement [7]. Partial transpose of density matrix (25) yields

$$\tau^T = \frac{1}{4}\begin{pmatrix} 4\alpha^2 n + (1-n)^2 & 0 & 0 & -2\alpha\sqrt{1-\alpha^2}(1-n^2) \\ 0 & 1-n^2 & 2\alpha\sqrt{1-\alpha^2}(1+n^2) & 0 \\ 0 & 2\alpha\sqrt{1-\alpha^2}(1+n^2) & 1-n^2 & 0 \\ -2\alpha\sqrt{1-\alpha^2}(1-n^2) & 0 & 0 & -4\alpha^2 n + (1+n)^2 \end{pmatrix}.$$

It is possible diagonalize τ^T and get it's eigenvalues

$$\lambda_1 = \frac{1}{4}(1-n^2) + \frac{1}{2}\alpha\sqrt{1-\alpha^2}(1+n^2), \tag{26}$$

$$\lambda_2 = \frac{1}{4}(1-n^2) - \frac{1}{2}\alpha\sqrt{1-\alpha^2}(1+n^2), \tag{27}$$

$$\lambda_3 = \frac{1}{4}(1+n^2) + \frac{1}{2}\sqrt{n^2 + \alpha^2(1-\alpha^2)(n^4 - 6n^2 + 1)}, \tag{28}$$

$$\lambda_4 = \frac{1}{4}(1+n^2) - \frac{1}{2}\sqrt{n^2 + \alpha^2(1-\alpha^2)(n^4 - 6n^2 + 1)}. \tag{29}$$

For $0 < n, \alpha < 1$ eigenvalues λ_1, λ_3 and λ_4 are always positive. The eigenvalue λ_2 is negative for $\alpha\sqrt{1-\alpha^2} > R$ where

$$R = \frac{1-n^2}{2(1+n^2)}. \tag{30}$$

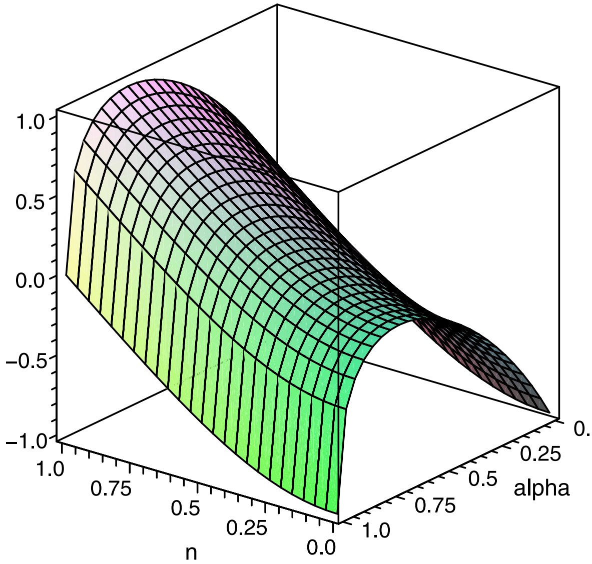

Fig. 1 Plot of negativity versus n and α

In this range the logarithmic negativity takes the form

$$N = \log_2 \left\{ \frac{1}{2}(1+n^2)\left(1+2\alpha\sqrt{1-\alpha^2}\right) \right\}. \tag{31}$$

In ultra relativistic limit $n \to 0$: $N \to \log_2\{\frac{1}{2} + \alpha\sqrt{1-\alpha^2}\}$, then the state does not have distillable entanglement for any α. For the rest frame $n = 1$: $N = \log_2\{1 + 2\alpha\sqrt{1-\alpha^2}\}$. In the range $0 < \alpha < 1/\sqrt{2}$ the larger α, the stronger the initial entanglement; but in the range $1/\sqrt{2} < \alpha < 1$ the larger α, the weaker the initial entanglement. For finite velocity, the monotonic decrease of N with increasing boost speed for different α means that the entanglement of the initial state is lost due to Wigner rotation. From Fig. 1 it is found that the entanglement in moving frame, for α and it's normalized partner $\sqrt{1-\alpha^2}$, will be degraded as boost speed increases. Here we calculate the concurrence which is defined as follows

$$C = 2\sqrt{\det \rho_A} = 2\sqrt{\det \rho_B}. \tag{32}$$

For density matrix (25) we have

$$\rho_A = \rho_B = \begin{pmatrix} \alpha^2 n + (1-n)/2 & 0 \\ 0 & -\alpha^2 n + (1+n)/2 \end{pmatrix}, \tag{33}$$

then

$$C = \frac{1}{2}\sqrt{(1-n+2\alpha^2 n)(1+n-2\alpha^2 n)}. \tag{34}$$

In non relativistic limit ($n = 1$) we have: $C = 2\alpha\sqrt{1-\alpha^2}$ and in ultrarelativistic limit ($n = 0$): $C = 1/2$. It is interesting that for maximally entangled state as $\alpha = 1/\sqrt{2}$ for all values of n concurrence is 1.

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